Killara HIGH SCHOOL

Student Number

2016

Mathematics Extension 2

Trial HSC

Date of Task : 3rd August 2016

General Instructions

- Reading time – 5 minutes
- Working time **3** hours
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- In Questions 11 to 16 show relevant mathematical reasoning and/or calculations
- Answer each question in a separate writing booklet
- This paper must not be removed from the examination room
- A reference sheet is provided at the back of this paper
- Diagrams are NOT to scale

Total Marks – 100

Section I - Pages 2 - 4

10 marks

- Attempt Questions 1 to 10
- Allow about 15 minutes for this section

Section II - Pages 5 -15

90 marks

- Attempt Questions 11 to 16
- Allow about 2 hours 45 minutes for this section

	Marks
Multiple choice	**/ 10**
Q11	**/ 15**
Q12	**/ 15**
Q13	**/ 15**
Q14	**/ 15**
Q15	**/ 15**
Q16	**/ 15**

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 40% of the HSC *Course Assessment*

Section I

10 marks
Attempt Questions 1 to 10
Allow about 15 minutes for this section

The multiple-choice answer sheet for questions 1 to 10(Detach from paper)

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1. Which of the following statements is always correct?

(A) If $z = a + ib$ is in the first quadrant, then arg $(z) = tan^{-1}\left(-\frac{b}{a}\right)$.

(B) If $z = a + ib$ is in the second quadrant, then arg $(z) = tan^{-1}\left(\frac{b}{a}\right)$.

(C) If $z = a + ib$ is in the fourth quadrant, then arg $(z) = tan^{-1}\left(\frac{b}{a}\right)$.

(D) If $z = a + ib$ is in the third quadrant, then arg $(z) = tan^{-1}\left(\frac{b}{a}\right)$.

2. What are the values of real numbers p and q such that $1 - i$ is a root of the equation $z^3 + pz + q = 0$.

(A) $p = -2$ and $q = 4$.

(B) $p = 2$ and $q = 4$.

(C) $p = 2$ and $q = -4$.

(D) $p = -2$ and $q = -4$.

3. Let ω be a complex root such that $\omega^n = 1, \omega \neq 1$.
Find the value of $\sum_{k=0}^{n}\left(\omega^k + \frac{1}{\omega^k}\right)$.

(A) 0

(B) 1

(C) 2

(D) 3

4. Which of the following statements is not necessarily true?

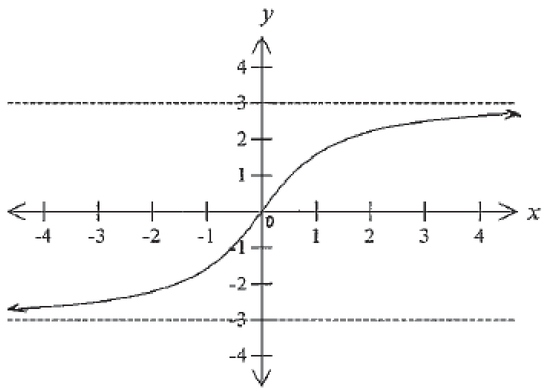
(A) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(B) If a polynomial has a root of multiplicity *n*, then the polynomial has degree *n*.

(C) If $f(x) < g(x)$ for $0 \le x \le a$ then $\int_0^a f(x)\,dx < \int_0^a g(x)\,dx$

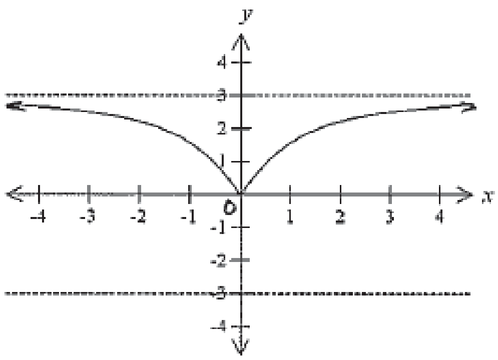
(D) The expression $z^n = 1$ has exactly $n - 1$ non-real roots, if *n* is odd.

5. The diagram shows the graph of the function *f(x)*.

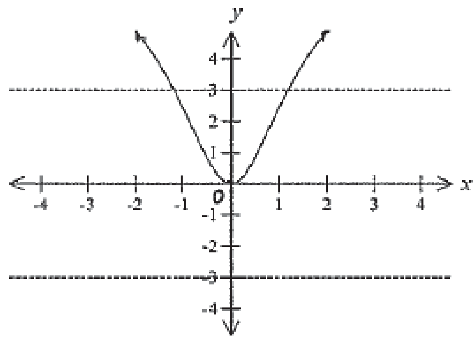


Which of the following graph is the graph of $y = \sqrt{f(x)}$

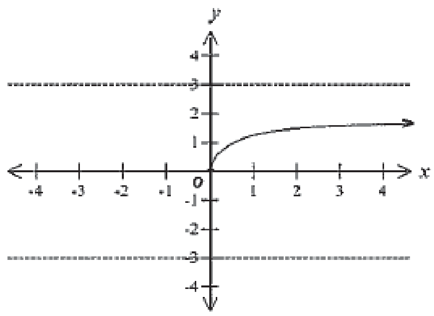
(A)



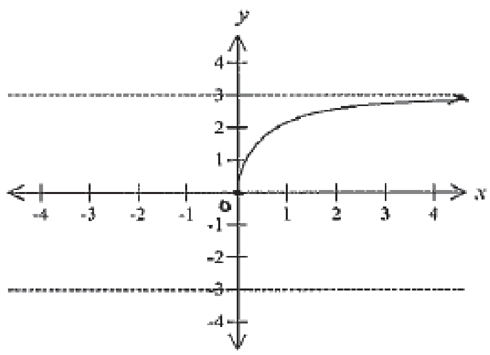
(B)



(C)



(D



6. The letters of the word UBRUTUS are arranged in a line. In how many of these arrangements are all U's separated? (i.e. No U can be next to another U, e.g. BURUTUS)

(A) 10

(B) 72

(C) 240

(D) 24

7. The circle $x^2 + y^2 = 4$ is rotated about the line $x = 3$. Using the washer method (annuli), the volume V of the solid generated is given by,

(A) $2\pi \int_0^2 \left[\left(3 + \sqrt{4 - y^2}\right)^2 - \left(3 - \sqrt{4 - y^2}\right)^2\right] dy$

(B) $\pi \int_0^2 \left[\left(3 + \sqrt{4 - y^2}\right)^2 - \left(3 - \sqrt{4 - y^2}\right)^2\right] dy$

(C) $2\pi \int_0^2 [(\sqrt{4 - y^2} - 9)^2] dy$

(D) $\pi \int_0^2 [(9 - \sqrt{4 - y^2})^2] dy$

8. The solution to $\frac{x(x-5)}{4-x} < -3$ is:

(A) $x < 0,\ 4 < x < 5$

(B) $x > 5, 0 < x < 4$

(C) $x < 2, 4 < x < 6$

(D) $x > 6, 2 < x < 4$

9. Given the hyperbola $\frac{x^2}{144} - \frac{y^2}{25} = 1$ then:

(A) eccentricity $e = \frac{13}{12}$ and foci are at $\left(\pm\frac{144}{13}, 0\right)$

(B) eccentricity $e = \frac{13}{5}$ and foci are at $(\pm 13, 0)$

(C) eccentricity $e = \frac{13}{12}$ and foci are at $(\pm 13, 0)$

(D) eccentricity $e = \frac{13}{5}$ and foci are at $\left(\pm\frac{144}{13}, 0\right)$

10. Suppose *f(x)* is a continuous smooth function over $a \le x \le b$ and *g(x)* is a continuous smooth function over $c \le x \le d$. Which of the following integrals is always greater than or equal to the other choices?

(A) $\int_a^b f(x)\,dx + \int_c^d g(x)\,dx$

(B) $\int_a^b |f(x)|\,dx + \int_c^d |g(x)|\,dx$

(C) $\left|\int_a^b f(x)\,dx + \int_c^d g(x)\,dx\right|$

(D) $\left|\int_a^b f(x)\right|dx + \left|\int_c^d g(x)\right|dx$

Section II

90 marks
Attempt Questions 11 to 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Find $\int \sin x \cos x .\, e^{cos2x} dx$ 1

(b) i) Split into partial fractions: $\dfrac{8}{(x+2)(x^2+4)}$ 2

ii) Hence evaluate: $\displaystyle\int_0^2 \frac{8}{(x+2)(x^2+4)}\, dx$ 3

(c) Use the substitution $x = \sin\theta$ to find $\displaystyle\int \frac{\sqrt{1-x^2}}{x}\, dx$ 3

(d) By using the substitution $t = \tan\dfrac{x}{2}$, evaluate $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1+\sin x - \cos x}$ 4

(e) The complex numbers z and ω are such that $z = \frac{3a-5i}{1+2i}$ and $\omega = 1 - 13bi,$ where a and b are real numbers. 2

Given that $\bar{z} = \omega$, where $\bar{z}$ is the complex conjugate of z, find the values of a and b

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

(a) Consider the hyperbolas $H_1 : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $H_2 : \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$. 2

Show that the foci of both hyperbolas lie on the same circle.

(b) i) On **<u>one</u>** Argand diagram, shade the region satisfying the following inequalities:

$$|z + 1 - 3i| \leq 2$$ 2

$$\frac{2\pi}{3} \leq \arg(z - 2i) \leq \frac{3\pi}{4}$$ 2

and $|z| \geq |z + 2|$ 2

Label each locus clearly.

ii) Express z, satisfying the above inequalities, in the form $a + ib$ when $Re(z)$ takes its minimum value. 2

(c) i) Using de Moivre's theorem, show that $\cos 5\theta = sin^5\theta\,(t^5 - 10t^3 + 5t)$, where $t = \cot\theta$. 2

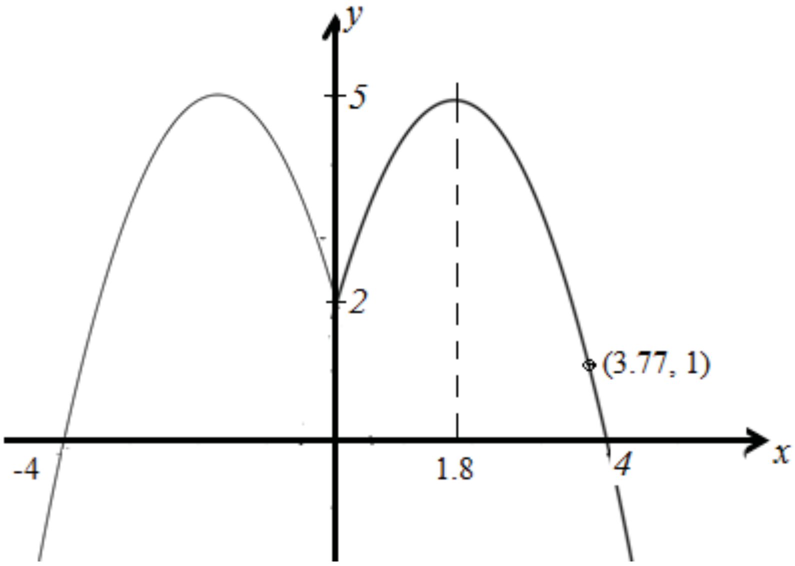
ii) Show that $cot^2\left(\frac{\pi}{10}\right)$ is a root of the equation $x^2 - 10x + 5 = 0$ 2

iii) Hence find the exact value of $cot^2\left(\frac{\pi}{10}\right)$ 1

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) The sketch is of the even function $y = f(x)$



On separate number planes, sketch each of the following. Clearly showing all important features.

i) $y^2 = f(x)$ 2

ii) $y = \dfrac{1}{f(x)}$ 2

iii) $y = x.f(x)$ 2

iv) $y = ln(f(x))$ 2

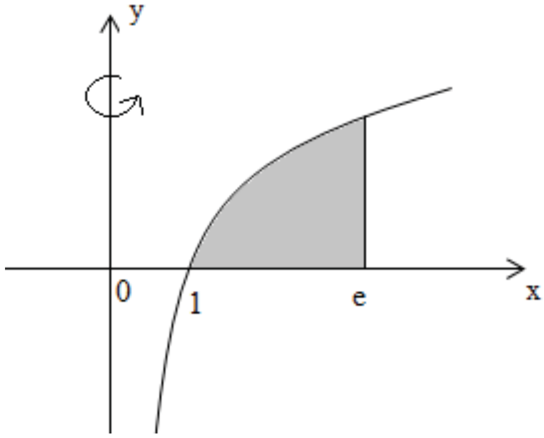
(b) Show that, if $x^3 + px + r = 0$ has a root of multiplicity two, then $27r^2 + 4p^3 = 0$ 3

Question 13 continues on page 9

Question 13 (continues)

(c) i) Show that $\int_1^e x\ln x \, dx = \frac{1}{4}(e^2+1)$ 2

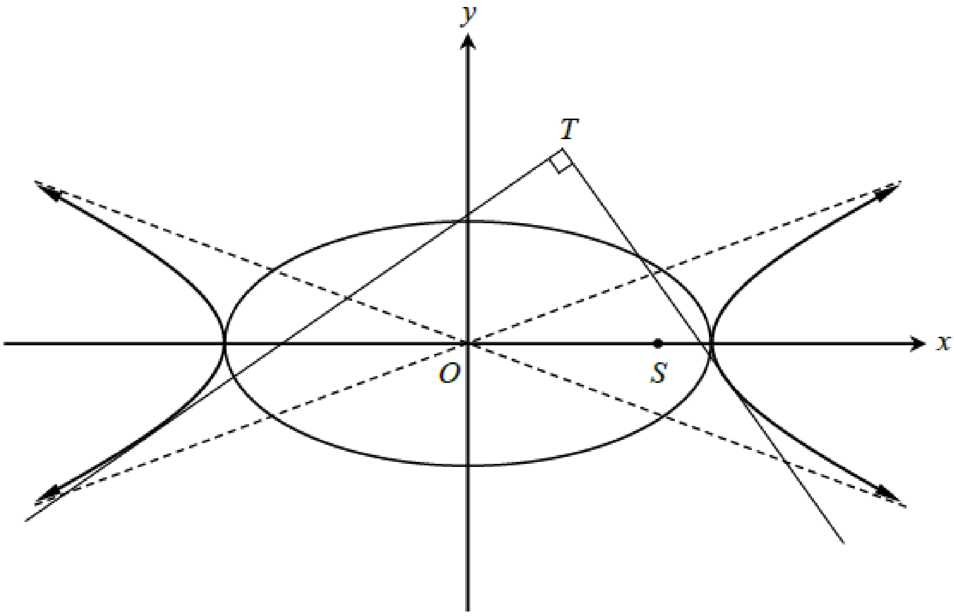
ii) The region bounded by $y = lnx$, $x = e$ and the x-axis is rotated about the y-axis. Using the method of cylindrical shells, find the volume of rotation. 2



End of question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its corresponding hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where $a > b$, on the same set of axes. Let the positive focus of the ellipse be S. From two points on the hyperbola, mutually perpendicular tangents are drawn and intersect each other at T.



i. Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, is **3**

$$y = mx \pm \sqrt{a^2m^2 - b^2} \text{ for all values of } m$$

ii. Hence show that $m^2(a^2 - x^2) + 2mxy - (b^2 + y^2) = 0$ **1**

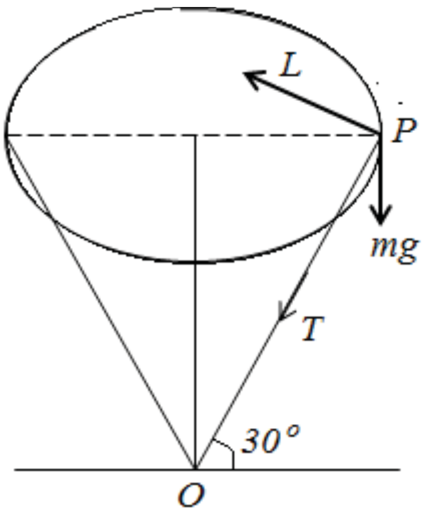
iii. Show that the locus of T is the circle $x^2 + y^2 = a^2 - b^2$ **1**

iv. Deduce that the triangle OTS is isosceles. **2**

Question 14 continues on page 11

Question 14 (continued)

(b)



A model plane of mass 5kg attached to the end of an inelastic wire of length 20m flies in a horizontal circle of elevation 30^o, while the other end of the wire is held fixed. The lift (force) L acts at right angles to the wire and L is twice the weight of the plane and $g = 9.8\text{m/s}^2$.

i) Draw a diagram to represent all the forces acting on the particle in the horizontal and vertical direction. 1

Hence find:

ii) the tension in the wire in Newtons. 2

iii) the speed of the plane in m/s 2

(c)

α and β are the roots of the equation $x^2 + px + q = 0$.

i) If $S_n = \alpha^n + \beta^n$,

Show that $S_{2n} = {S_n}^2 - 2q^n$ and $S_{2n+1} = S_n S_{n+1} + pq^n$ 3

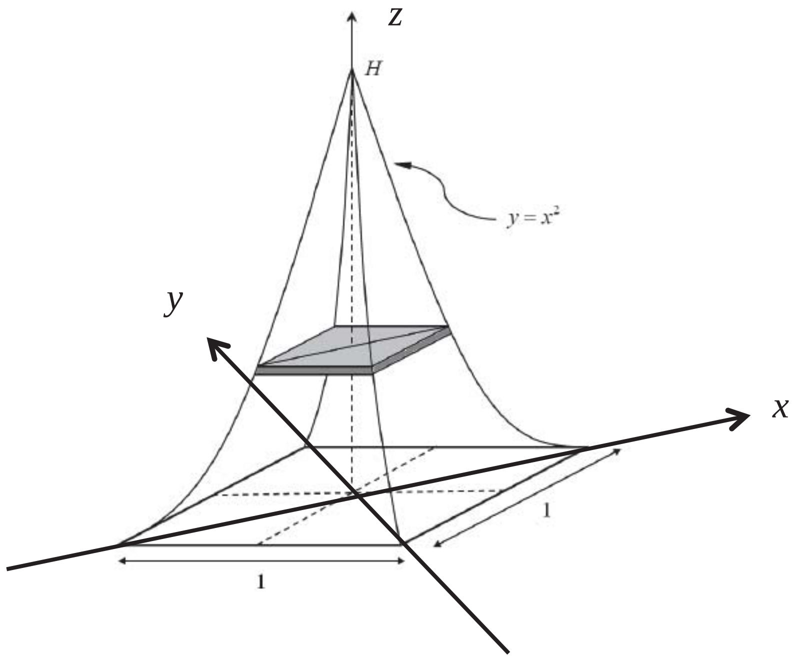
ii) Hence express S_5 in terms of p and q. 2

End of Question 14

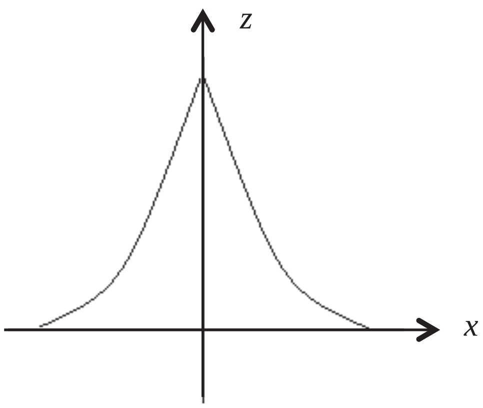
Question 15 (15 marks) Use a SEPARATE writing booklet

(a) A pyramid-like structure with curved edges has a square base of unit length. Cross sections taken parallel to the base are squares, and the 'pyramid' eventually ends at the tip with some height H. All the curved edges follow the shape of the curve $y = x^2$, with the corners of the base being the vertex of the parabola.

Let the height, from the base, of an arbitrary slice be h.



A vertical cut taken through the middle of the pyramid is shown in the diagram below.



i) What is the equation of the curve $y = x^2$ relative to the x and z-axes shown? 1

Question 15 continues on page 12

Question 15 (continued)

ii) Show that the length of the diagonal of the slice is **2**

$$d = \sqrt{2}\,(1 - \sqrt{2h}).$$

iii) Show that $H = \frac{1}{2}$ **1**

iv) Hence find the volume of the solid. **2**

(b) Let $f(x) = \dfrac{(x-2)(x+1)}{5-x}$ for $x \neq 5$.

i) Show that $f(x) = -x - 4 + \dfrac{18}{5-x}$. **1**

ii) Sketch the curve $y = f(x)$. Label all the asymptotes, and show the x intercepts. (There is no need to find the stationary points). **3**

iii) Hence find the values of x for which $f(x)$ is positive and the values of x for which $f(x)$ is negative. **1**

(c) A jar contains w white jelly beans and r red jelly beans. Three jelly beans are taken at random from the jar and eaten.

i) Write down an expression, in terms of w and r, for the probability that these 3 jelly beans were white. **1**

Garry observed that if the jar had initially contained $(w + 1)$ white and r red jelly beans, then the probability that the 3 eaten jelly beans were white would have been double that in part (i).

ii) Show that $r = \dfrac{w^2 - w - 2}{5 - w}$. **2**

iii) Using part (b) (iii), or otherwise, determine all possible numbers of white and red jelly beans. **1**

End of question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) If $I_n = \int_0^1 x(1-x^3)^n dx$,

i) Prove that $I_n = \frac{3n}{3n+2} I_{n-1}$ 3

ii) Hence find the value of $\int_0^1 x\,(1-x^3)^4 dx$ 2

(b) A particle P of mass $m\ kg$ projected vertically upwards from the ground with initial velocity $U\ ms^{-1}$experiences air resistance of mkv^2, where k is a positive constant and v is its velocity. The greatest height H that it will attain is given by, $H = \frac{1}{2k} log_e\left(1 + \frac{U^2}{{V_T}^2}\right)$, where V_T is the terminal velocity on its downward fall. The air resistance that it experiences on its downward motion is also mkv^2. Acceleration due to gravity is $g\ ms^{-2}$.

i) Write down the equation of motion during its downward motion. 1

ii) Express its terminal velocity in terms of k and g. 1

iii) Show that the distance travelled on its return to the point of projection x. is given by 2

$$x = -\frac{1}{2k}\log_e\left(1 - \frac{v^2}{V_T^2}\right)$$

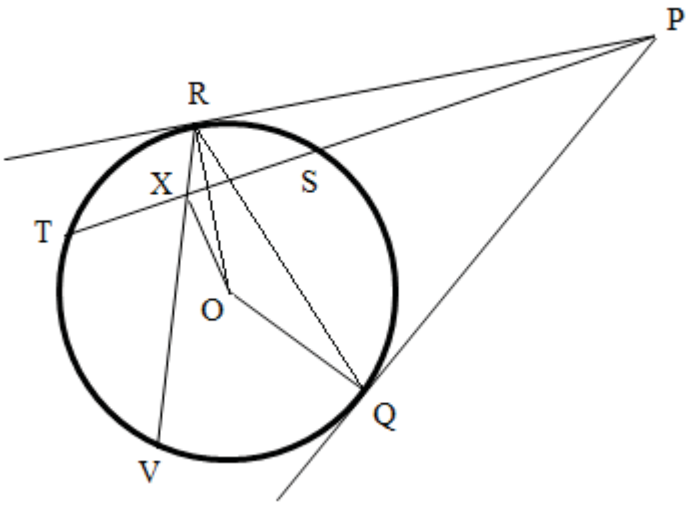
iv) Show that it returns to the ground with speed W, where $W^{-2} = U^{-2} + {V_T}^{-2}$. 1

Question 16 continues on page 14

Question 16 (continued)

(c)

In the diagram, O is the centre of the circle. From a point P, tangents are drawn to the circle touching the circle at Q and R. A line through P cuts the circle at S and T and OX bisects the chord ST. RX produced cuts the circle at V.



i) Prove that $ORPQ$ and $OXRP$ are cyclic quadrilaterals. 2

ii) Prove that $TS \parallel VQ$. 3

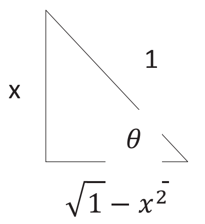
End of Paper

2016 X2 Trial-Solution

1. D 2. A 3. C 4. B 5. C 6. C 7. A 8. D 9. C 10. B

Q11

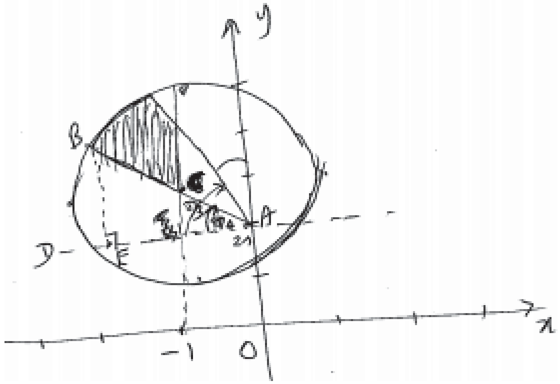
(a)		
$\int \sin x \cos c.\, e^{cos2x} dx = \; -\frac{1}{4}\int 2\sin x \cos c.\, e^{cos2x} dx$ $= \; -\frac{1}{4} e^{cos2x} + C$	1 correct answer	
(b) $\frac{8}{(x+2)(x^2+4)} = \frac{a}{x+2} + \frac{bx+c}{x^2+4}$ $(x^2 + 4)a + (x + 2)(bx + c) = 8$ Constant term: $4a + 2c = 8$ (1) Let $x = -2$ $8a = 8$ (2) $\therefore$ a = 1 and c = 2 Coefficient of x : $2b + c = 0 \Rightarrow b = -1$	2 correct values of a, b & c	
$\int_0^2 \frac{8}{(x+2)(x^2+4)} = \int_0^2 \frac{1}{x+2} + \frac{2-x}{x^2+4}\, dx$ $= \left[\ln(x+2\right]_0^2 + \int_0^2 \frac{2}{x^2+4} - \frac{1}{2}\int_0^2 \frac{2x}{x^2+4} dx$	1 correct expression of integrands	
$= \left[\ln(x+2) + \tan^{-1}\frac{x}{2} - \frac{1}{2}\ln(x^2+4)\right]_0^2 = \ln\sqrt{2} + \frac{\pi}{4}$	1 correct integration 1 correct answer	

(c) x = sinθ dx = cosθ dθ 	1 correct changing variables	
$\int \frac{\sqrt{1-x^2}}{x}\,dx = \int \frac{\cos\theta}{\sin\theta}\cos\theta\,d\theta$ $= \int \frac{cosec\theta\,(cosec\theta+cot\theta)}{cosec\theta+\cot\theta} - sinx\ d\theta$ $= -\,ln(\text{cosec}\theta + cot\theta) + cosx + C$	1 correct integration	
$= -\ \ln\left(\frac{1}{x}+\frac{\sqrt{1-x^2}}{x}\right)+\sqrt{1-x^2}+C$	1 correct answer in term of x	
(d) $t=\tan\frac{x}{2}$ $x=\frac{\pi}{2}\quad t=1$ $dt=\frac{1}{2}(1+t^2)dx$ $x=\frac{\pi}{3}\quad t=\frac{1}{\sqrt{3}}$		
$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1+\sin x-\cos x} = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$ $= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{dt}{1+t^2+2t-1+t^2}$	1 correct substitution	
$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{dt}{t(t+1)} = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{t}-\frac{1}{t+1}dt$	1 correct integrands	

$= \left[\ln(t) - \ln(t+1)\right]_{\frac{1}{\sqrt{3}}}^{1}$	1 correct integration	
$= = -\ln 2 - \ln\left(\frac{1}{\sqrt{3}}\right) + \ln\left(\frac{1}{\sqrt{3}} + 1\right)$		
$= \ln\left(\frac{1+\frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}}\right) = \ln\left(\frac{\sqrt{3}+1}{2}\right)$	1 correct answer	
(e)		
$z = \frac{3a-5i}{1+2i} \times \frac{1-2i}{1-2i}$		
$= \frac{(3a-10)-(6a+5)i}{5}$		
$\bar{z} = \frac{3a-10}{5} + \frac{(6a+5)i}{5} = 1-13bi$	1 correct answer of $\bar{z}$	
Equate real & im. parts		
$\frac{3a-10}{5} = 1 \quad \Rightarrow \quad a = 5$		
$\frac{6a+5}{5} = -13b \quad \Rightarrow \quad \frac{30+5}{5} = -13b$		
$b = -\frac{7}{13}$	1 correct answers of a & b	

Q12

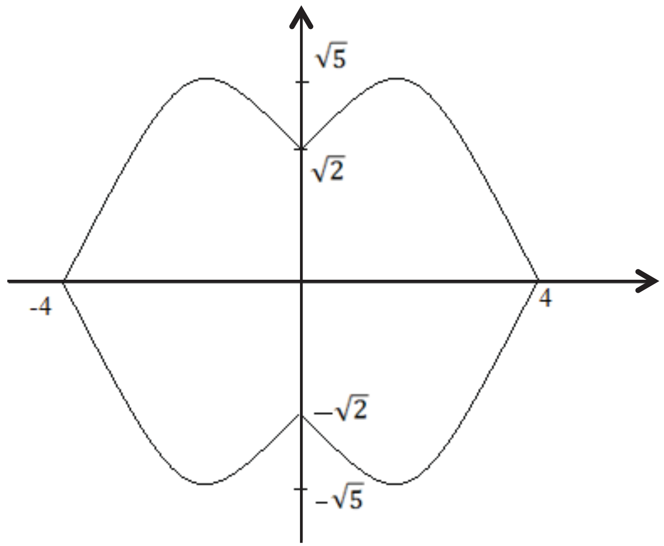
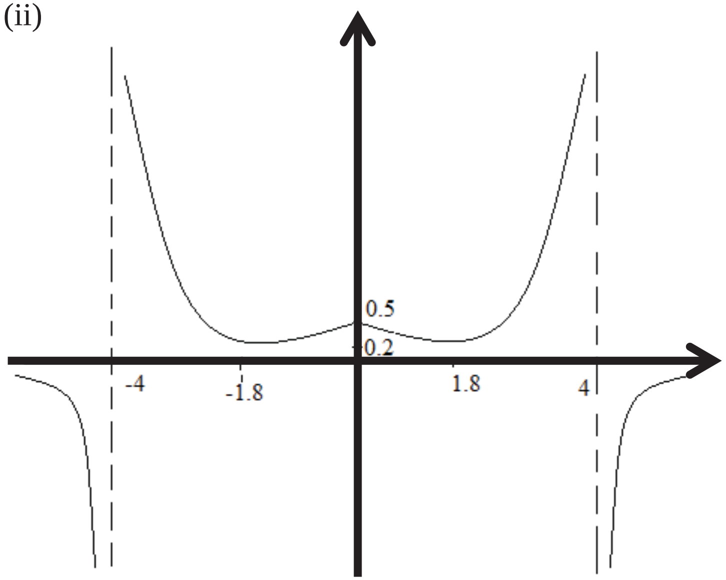
<table>
<tr><td>(a) $H_1: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $H_2: \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
Foci of H_1: $(\pm ae_1, 0)$ $a^2e_1^2 = a^2 + b^2$
Foci of H_2: $(0, \pm be_2)$ $b^2e_2^2 = a^2 + b^2$
$= (0, \pm ae_1)$ $\therefore be_2 = ae_1$
$\therefore$ The 4 foci are ae_1 from the centre
$\therefore$ all on the same circle with radius ae_1.</td><td>1 correct foci

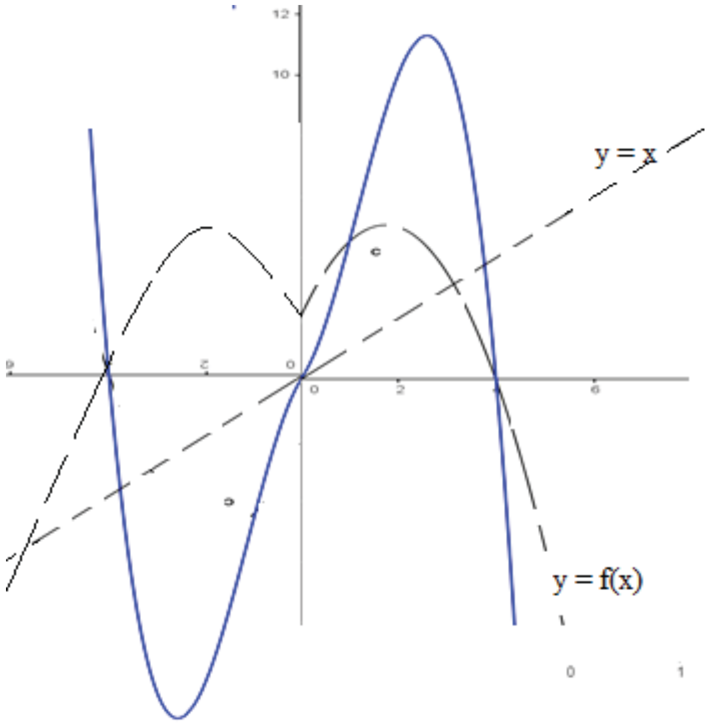
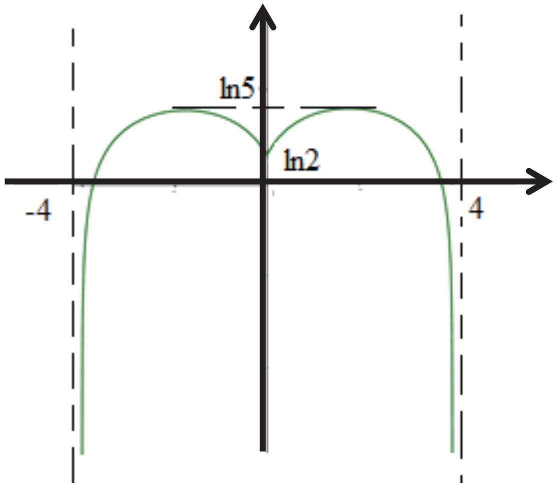
1 correct conclusion</td><td></td></tr>
<tr><td>(b) (i) let $z = x + iy$
$|iz + 3 + i| \le 2 \Rightarrow (3-y)^2 + (x+1)^2 \le 4$
$|z| \ge |z+2| \Rightarrow x \le -1$

(ii) when $Re(z)$ is minimum
z is represented by B.
$AC = \sqrt{2}$, $CB = 2$
$\therefore AB = 2 + \sqrt{2}$
$\therefore Re(z) = AE$
$= AB \cdot \cos \pi/4$
$= \sqrt{2} + 1$
$BE = \sqrt{2} + 1$
$\therefore z = (\sqrt{2}+1) + i(3 + \sqrt{2})$</td><td>2 for each correct graph

2 correct answer for z</td><td></td></tr>
</table>

12) (c) (i) $(\cos\theta + i\sin\theta)^5 = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$ $+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$ Equating the real parts, $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $\therefore \frac{\cos 5\theta}{\sin^5\theta} = \cot^5\theta - 10\cot^3\theta + 5\cot\theta$ $= t^5 - 10t^3 + 5t$ $\Rightarrow \cos 5\theta = \sin^5\theta\,(t^5 - 10t^3 + 5t)$ (ii) when $\theta = \frac{\pi}{10}$, $0 = \sin^5\left(\frac{\pi}{10}\right)\left[t^5 - 10t^3 + 5t\right]$ $\Rightarrow t(t^4 - 10t^2 + 5) = 0$ $t = \cot^2\left(\frac{\pi}{10}\right)$ $\left[\left(\cot^2\left(\frac{\pi}{10}\right)\right)^2 - 10\left(\cot^2\left(\frac{\pi}{10}\right)\right) + 5\right] = 0$ $\Rightarrow \cot^2\left(\frac{\pi}{10}\right)$ is a root of $x^2 - 10x + 5 = 0$. (iii) $\cot^2\left(\frac{\pi}{10}\right) = 5 \pm \sqrt{20}$ $\cot^2\left(\frac{\pi}{10}\right) > 1$ $\Rightarrow \cot^2\left(\frac{\pi}{10}\right) = 5 + \sqrt{20}$	1 correct expansion 1 correct answer 2 correctly show $\cot^2\left(\frac{\pi}{10}\right)$ is a root 1 correct answer	

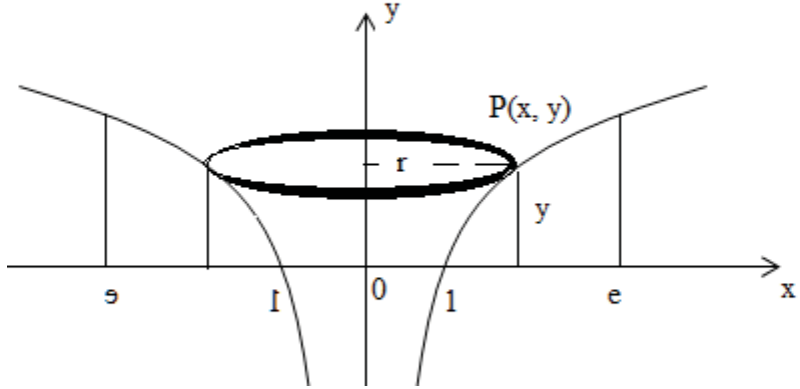
Q13

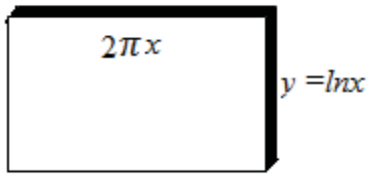
(a) (i) 	**2 for each correct graph with all important features**	
(ii) 	**2**	

(iii)  	**2**	
(iv)  	**2**	

(b) $\frac{d}{dx}f(x)=3x^2+p=0$ $\Rightarrow \quad x^2=-\frac{p}{3}$ Since f(x) has a root of multiplicity of 2, then $f\left(\sqrt{-\frac{p}{3}}\right)= f'\left(\sqrt{-\frac{p}{3}}\right)=0$ **<u>Method 1</u>** $\therefore f\left(\pm\sqrt{-\frac{p}{3}}\right)=\pm\sqrt{-\frac{p}{3}}\left(-\frac{p}{3}+1\right)=-r$ square both sides $=-\frac{p}{3}\left(-\frac{p^2}{9}-\frac{2p^2}{3}+P^2\right)=r^2$ $=-4p^3=27r^2 \quad \therefore \quad 4p^3+27r^2=0$ **<u>Method 2</u>** $x(x^2+p)=-r$ Square both sides $x^2(x^2+p)=r^2$ $sub \Rightarrow \quad x^2=-\frac{p}{3}$ $-\frac{p}{3}\left(-\frac{p}{3}+p\right)^2=r^2$ $-\frac{4p^3}{27}=r^2 \quad \Rightarrow \quad 27r^2+4p^3=0$ (c) (i) $\int_1^e x\ln x\, dx=\left[\frac{1}{2}x^2\ln x\right]_1^e-\frac{1}{2}\int_1^e\frac{x^2}{x}dx$ $=\frac{e^2}{2}-\left[\frac{x^2}{4}\right]_1^e$ $=\frac{e^2}{4}+\frac{1}{4}=\frac{1}{4}\left(e^2+1\right)$	**1 correct value of x** **1 correct sub. for x** **1 correct answer** **1 correct integration** **1 correct answer**	

(ii)





$\delta V = 2\pi x \ln x dx$

$V = \lim_{dx\to 0} \sum_{0}^{e} 2\pi x \ln x dx$

$= 2\pi\int_{0}^{e} x \ln x dx = 2\pi \times \frac{1}{4}(e^2+1)$

$= \frac{\pi}{2}\left(e^2+1\right)\ unit^3$

1 correct expression of V

1 correct answer

Q14

(a) (i) Let the general equation of the tangent be $y = mx \pm k$ (1) Equation of the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $b^2x^2 - a^2y^2 = a^2b^2$, sub into (1) for point of intersection. $b^2x^2 - a^2(m^2x^2 + 2mkx + k^2) = a^2b^2$ Since there is only one point of intersection $\Delta = 0$ $\therefore\ \Delta = 4m^2a^4k^2 - 4(b^2 - a^2m^2)(a^2k^2 - a^2b^2) = 0$ $= 4m^2a^4k^2 - 4b^2k^2a^2 - 4a^4m^2k^2 + 4a^2b^4 - 4a^4b^2m^2 = 0$ $4b^4k^2a^2 = -4a^2b^4 + 4a^4b^2m^2$ $k^2 = a^2m - b^2 \Rightarrow k = \pm\sqrt{a^2m - b^2}$ $\therefore y = mx \pm\sqrt{a^2m - b^2}$ **(ii)** $(y - mx)^2 = a^2m^2 - b^2$ $y^2 - 2myx + m^2x^2 = a^2m^2 - b^2$ $m^2x^2 - 2myx + y^2 + b^2 - a^2m^2 = 0$ $m^2(x^2 - a^2) - 2mxy + y^2 + b^2 = 0$ (1) **(iii)** At T, $m_1 \times m_2 = -1$	1 correct equation for points of intersection 1 correct expression of Δ 1 correct answer 1 correct expression Showing $m_1 \times m_2 = -1$ and	

Where $m_1 \times m_2$ is the product of roots of (1) | conclusion

$$\therefore \frac{y^2+b^2}{x^2-a^2} = -1 \quad \Rightarrow \quad y^2 + b^2 = a^2 - x^2$$
$$\Rightarrow \quad x^2 + y^2 = a^2 - b^2$$

Hence the locus of T is a circle with radius $a^2 - b^2$ and centre O.

(iii) The focus of the ellipse is (ae, 0)

1 finding OT

$OT = a^2 - b^2$ (radius)

1 finding OS with conclusion

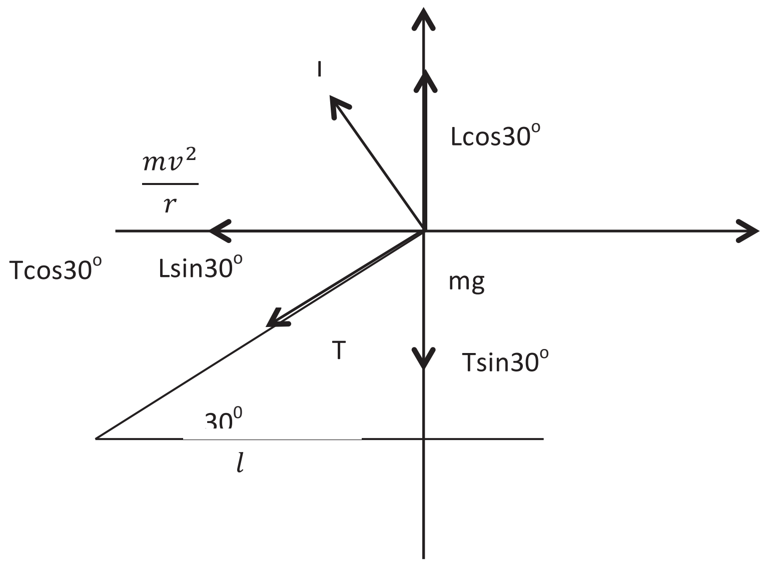
$$e = \frac{\sqrt{b^2 - a^2}}{a} \quad \Rightarrow \quad ae = \sqrt{b^2 - a^2}$$

$\therefore \quad OT = OS$

Hence OTS is an isosceles triangle. (2 equal sides).

(b)

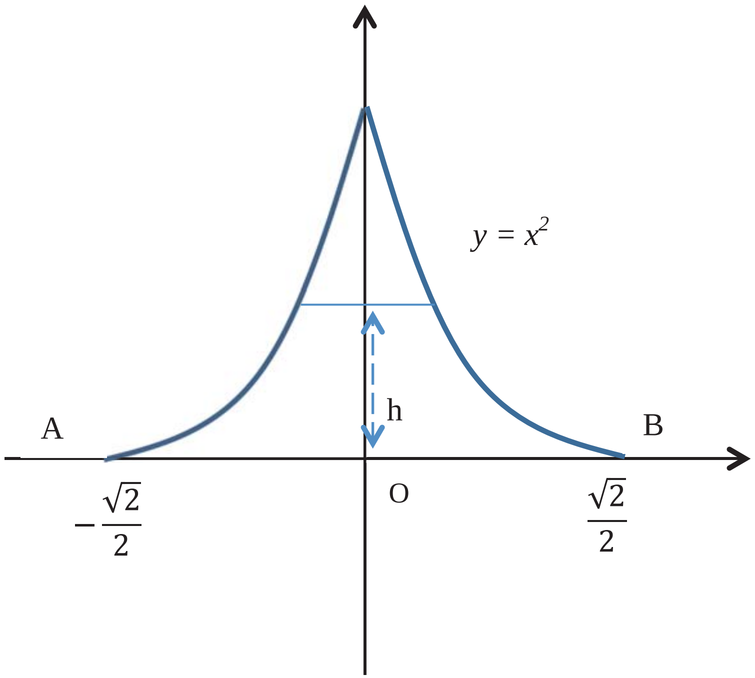
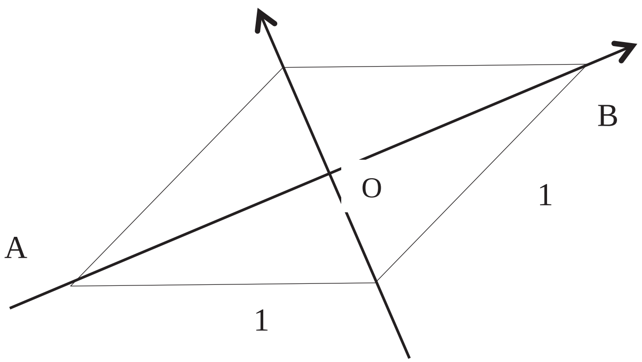
1 correct diagram with all the resultant forces



Resultant forces: L=2×5×9.8 = 98N Vertically: $r = 20cos30^o = 17.32$ $Lcos30^o = mg + Tsin30^o$ $98\times\frac{\sqrt{3}}{2} = 5\times9.8 + \frac{T}{2}$ $T = 2(84.87 - 49) = 71.74N$ Horizontally $\frac{mv^2}{r} = Lsin30^O + Tcos30^o$ $\frac{5v^2}{17.32} = 9.8\times\frac{1}{2} + \frac{\sqrt{3}}{2}T$ Sub T = 71.74N $\frac{5v^2}{17.32} = 49 + \frac{\sqrt{3}}{2}\times71.74$ $v^2 = 111.13\times17.32 \div 5 = 384.95$ $\therefore v = 19.62m/s$ (c) (i) $\alpha + \beta = -p,$ $\alpha\beta = q$ $S_n = \alpha^n + \beta^n$ $S_{2n} = \alpha^{2n} + \beta^{2n}$ $= (\alpha^n)^2 + (\beta^n)^2 = (\alpha^n + \beta^n)^2 - 2(\alpha\beta)^n$ $= S_n{}^2 - 2q^n$ (ii) $S_{2n+1} - S_nS_{n+1} = \alpha^{2n+1} + \beta^{2n+1} - (\alpha^n + \beta^n)(\alpha^{n+1} + \beta^{n+1})$ $= \alpha^{2n+1} + \beta^{2n+1} - \alpha^{2n+1} - \beta^{2n+1} - \alpha^n\beta^{n+1} - \beta^n\alpha^{n+1}$ $= -\alpha^n\beta^n(\alpha + \beta)$ $= -q^n\times -p = pq^n$ $\therefore S_{2n+1} = S_nS_{n+1} + pq^n$	1 Showing equation involving T 1 correct answer 1 showing equation involving v 1 correct answer for v 1 1 1	

(iii) $s_5 = s_{2\times2+1}$ where $n = 2$, Using result from (ii) $= s_2.s_3 + pq^2$ Where $s_2 = \alpha^2 + \beta^2 = p^2 - 2q$ and $s_3 = s_1.s_2 + pq$ $= -p(p^2 - 2q) + pq$ $= -p^3 + 3pq$ $\therefore s_5 = (p^2 - 2q)(-p^3 + 3pq) + pq^2$ $= -p^5 + 3p^3q + 2p^3q - 6pq^2 + pq^2$ $= 5p^3q - 5pq^2 - p^5$	 1 1	

Q15

  AB = $\sqrt{1+1} = \sqrt{2}$ OA = $\pm\frac{\sqrt{2}}{2}$ Equation of the curve y = x^2 is $z = \left(x \pm \frac{\sqrt{2}}{2}\right)^2$ (1)	1 correct equation	

When $z = h, x = \frac{d}{2}$ sub into (1) $\Rightarrow \quad h = \left(\frac{d}{2} - \frac{\sqrt{2}}{2}\right)^2 \qquad \Rightarrow \pm\sqrt{h} = \frac{d}{2} - \frac{\sqrt{2}}{2} \qquad as \quad d \le \sqrt{2}$ $\Rightarrow -\sqrt{h} = \frac{d}{2} - \frac{\sqrt{2}}{2}$ $\Rightarrow \frac{d}{2} = -\sqrt{h} + \frac{\sqrt{2}}{2}$ $\therefore \quad \frac{d}{2} = \frac{\sqrt{2}}{2}\left(1 - \sqrt{2h}\right)$ Hence the diagonal $d = \sqrt{2}\left(1 - \sqrt{2h}\right)$ (iii) When x = 0 sub into (1) $y = \left(0 \pm \frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$ (iv) Area of the cross-section is a rhombus: $A = \frac{d^2}{2}$ $= \frac{1}{2} \times \sqrt{2}\left(1 - \sqrt{2h}\right)\sqrt{2}\left(1 - \sqrt{2h}\right)$ $= (1 - 2\sqrt{2h} + 2h^2)$ $\delta V = (1 - 2\sqrt{2h} + 2h)dh$ $V = \lim_{\delta h \to 0} \sum_{0}^{\frac{1}{2}} \left(1 - 2\sqrt{2h} + 2h\right)\delta h$	1 correct sub. into equation in (i) 1 correct answer 1 1 correct expression of dv Must have this expression.	

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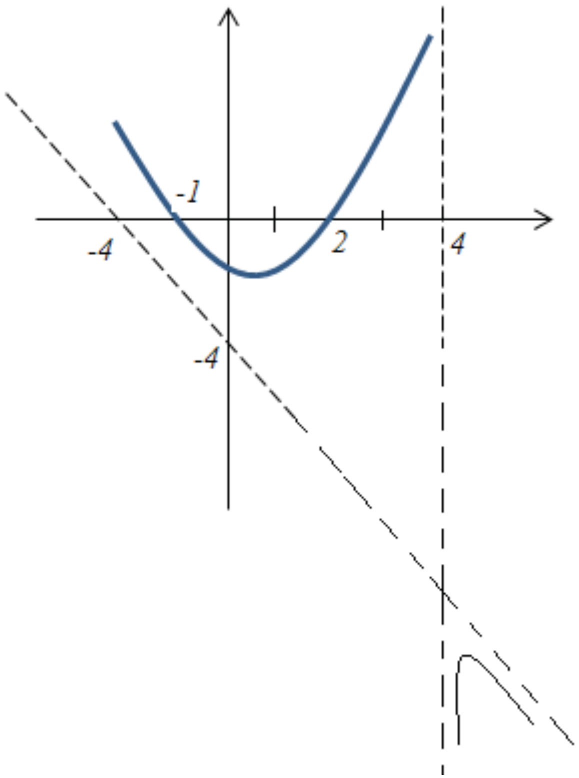
$$= \int_0^{0.5} 1-\sqrt{2}h^{\frac{1}{2}}+2h\ dh = \left[h-\frac{4}{3}h^{\frac{3}{2}}+h^2\right]_0^{0.5}$$

$$=\left(\frac{1}{2}-\frac{4\sqrt{2}}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}+\frac{1}{4}\right)=\frac{1}{12}$$

(b)

$- x - 4 + \frac{18}{5-x}$
$5 - x \,\overline{)\, x^2 - x - 2}$
$-5x + x^2$
$4x - 2$
$4x - 20$
18



(iii)
When x > 2 & x < -1, y > 0

When -1 < x < 2, y < 0.

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<td>1 correct answer

1

3 correct graph with all important features

1 correct answer</td>
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(c)

(i) $\frac{w}{w+r}.\frac{w}{w+r-1}.\frac{w-2}{w+r-2}$ (1)

$\frac{w+1}{w+r+1}.\frac{w}{w+r}.\frac{w-1}{w+r-1}$ (2)

If the jar had initially contained (w + 1) white and r red jelly beans, then

$\frac{w+1}{w+r+1}.\frac{w}{w+r}.\frac{w-1}{w+r-1} = 2(\frac{w}{w+r}.\frac{w}{w+r-1}.\frac{w-2}{w+r-2})$

$\Rightarrow \quad \frac{w+1}{w+r+1} = \frac{2w-4}{w+r-2}$

$(w+1)(w+r-2) = (2w-4)(w+r+1)$
$(w+1)r + (w+1)(w-2) = (2w-4)r + (2w-4)(w+1)$
$r(w+1-2w+4) = (w+1)(2w-4-w+2)$
$r(5-w) = (w+1)(w-2)$
$\therefore r = \frac{w^2-w-2}{5-w}$

(iii)

From the graph,
When w = 3, r = $\frac{w^2-w-2}{5-w}$

w = 4 r = $\frac{16-4-2}{1} = 10$

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1 correct expression

1 correct expression if there is 1 more white jelly bean.

1 correct answer

1 correct answer

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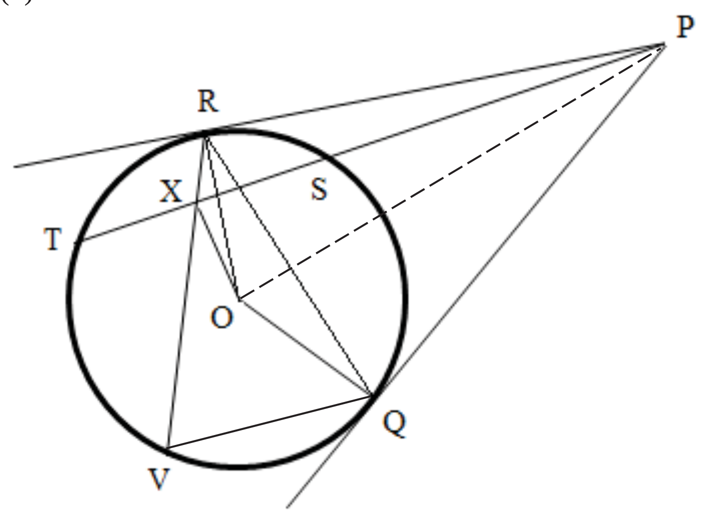
Q16

(i) $I_n = \int_0^1 x(1-x^3)^n dx,$		
$= \left[\frac{1}{2}x^2(1-x^3)\right]_0^1 - \frac{n}{2}\int_0^1 x^2.(-3x^2).(1-x^3)^{n-1}dx,$	1 correct first partial integration	
$= \frac{3n}{2}\int_0^1 (1-x^3-1).x.(1-x^3)^{n-1}\ dx$		
$= -\frac{3n}{2}\int_0^1 (1-x^3)^n + \frac{3n}{2}\int_0^1 x.(1-x^3)^{n-1}\ dx$	1 correct 2^{nd} partial integration	
$I_n + \frac{3n}{2}I_n = \frac{3n}{2}I_{n-1}$		
$\frac{3n+2}{2}I_n = \frac{3n}{2}I_{n-1}$	1 correct answer	
$\therefore \quad I_n = \frac{3n}{3n+2}I_{n-1}$		
(ii) $I_4 = \frac{12}{14}.\frac{9}{11}.\frac{6}{8}.\frac{3}{5}.I_0$ where $I_0 = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$	2 correct value of I_5 with working.	
$\therefore \quad I_4 = \frac{243}{1540}$		
(b) 		

$OR \perp RP$ and $OQ \perp PQ$ (tangent from an external point is perpendicular to radius at point of contact) **(1)**	1	
$\therefore \angle ORP + \angle OQP = 180^o$		
Hence ORPQ is cyclic quad. (Opposite angles are supplementary)		
X is the midpoint of TS, hence OX $\perp$ TS (radius is the perpendicular bisector of the chord TS)		
$\therefore \angle OXP = 90^o$		
$\angle ORP = 90^o$ from (1)		
$\therefore \angle OXP = \angle ORP$ (angles at the circumference subtend the same arc OP)	1	
Hence OXRP is cyclic quad.		
(ii)		
$\angle ROP = \angle RXP = \theta$ (angles at the circumference subtend the same arc PR in quad. OXRP) (2)		
$\angle TXV = \angle RXP = \theta$ (vertically opposite angles)	1	
RP = PQ (tangents to the circle from an external point)		
$\angle ROP = \angle QOP = \theta$ (angles at circumference subtend equal arcs RP and PQ), from (2)	1	
$\therefore \angle ROP + \angle QOP = 2\theta$		
$\angle RVQ = \theta$ (angle at the circumference is half the angle at the centre subtend the same arc RQ)		
$\angle TXV = \angle RVQ =$ (alternate angles are equal)	1	
$\therefore$ TX // VQ		
(iii) Downward motion		
		
(a) Equation of motion: $\ddot{x} = g - kv^2$	1	

(b) Teminal velocity happens when $\ddot{x} \to 0$ $\quad \therefore g = kv^2 \quad \Rightarrow \quad v = \sqrt{\frac{g}{k}}$	1	
(c)		
$v\frac{dv}{dx} = g - kv^2$		
$\frac{dx}{dv} = \frac{v}{g - kv^2}$		
$-2kdx = \frac{-2kv}{g - kv^2}$		
Integrate both sides		
$-2kx = \ln\left(g - kv^2\right) + C$	1	
$t = 0, x = 0 \quad v = 0$		
$\Rightarrow \; C = -\ln(g)$		
$\therefore x = -\frac{1}{2k}\ln\left(\frac{g - kv^2}{g}\right) = -\frac{1}{2k}\ln\left(1 - \frac{kv^2}{g}\right) \qquad (1)$		
When $V_T = \sqrt{\frac{g}{k}}$, sub into (1)		
$\Rightarrow x = -\frac{1}{2k}\log_e\left(1 - \frac{v^2}{V_T^2}\right)$	1	
The particle returns to the ground when x = max height, ie. H = x		
$\Rightarrow \quad \frac{1}{2k}\log_e\left(1 + \frac{U^2}{V_T^2}\right) = -\frac{1}{2k}log_e\left(1 - \frac{v^2}{{V_T}^2}\right),$		
$\Rightarrow \quad \left(1 + \frac{U^2}{V_T^2}\right) = \left(1 - \frac{v^2}{V_T^2}\right)^{-1} \quad \Rightarrow \quad \frac{V_T^2 + U^2}{V_T^2} = \frac{V_T^2}{V_T^2 - v^2}$		
$\Rightarrow \quad V_T^4 + V_T^2U^2 - v^2(V_T^2 + U^2) = V_T^4$		
$\therefore \quad v^2 = \frac{V_T^2U^2}{V_T^2 + U^2} \quad \Rightarrow \quad \frac{1}{v^2} = \frac{V_T^2 + U^2}{V_T^2U^2} = \frac{1}{U^2} + \frac{1}{V_T^2}$	1	